Math 10A
Worksheet, Discussion #11; Monday, 7/2/2018
Instructor name: Roy Zhao

1 Taylor Series

1.1 Concepts

1. The Taylor series for a function $f(x)$ around a point $x = c$ is given by

$$f(x) \approx f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \frac{f^{(4)}(c)}{4!}(x-c)^4 + \cdots$$

1.2 Problems

2. Use the second order Taylor series to approximate $\sqrt{17}$.

Solution: The formula for the second order Taylor series expanded at $x = c$ is

$$f(x) \approx f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2.$$

The closest square is 16, so we can expand around there since we know $\sqrt{16} = 4$. We have that $f(x) = \sqrt{x}, f'(x) = \frac{1}{2\sqrt{x}}, f''(x) = \frac{-1}{4x\sqrt{x}}$. Plugging in $c = 16$, we have that

$$f(x) \approx 4 + \frac{x-16}{8} - \frac{1}{512}(x-16)^2.$$

Plugging in $x = 17$, we have that

$$\sqrt{17} \approx 4 + \frac{1}{8} - \frac{1}{512} \approx 4.123.$$

3. Find the Taylor series for $x^5 + 3x^3 + 2x + 10$.

Solution: Taylor series give you a polynomial approximation for your function. But if your function is already a polynomial, then it gives the same thing. Try it out and verify it by yourself! So the Taylor series is just $x^5 + 3x^3 + 2x + 10$.

4. Use the second order approximation to $\sqrt[3]{28}$.

Solution: A close cube that we know is $3^3 = 27$. So we calculate the second order Taylor series expanded at $x = 27$ to get

$$\sqrt[3]{x} \approx 3 + \frac{x-27}{27} - \frac{(x-27)^2}{2187}.$$

So plugging in 28 gives

$$\sqrt[3]{28} \approx 3 + \frac{1}{27} - \frac{1}{2187} \approx 3.036.$$

5. Use the second order approximation to find $\ln 1.1$.

Solution: We know that $\ln 1 = 1$. So we can expand out at $x = 1$ to get

$$\ln x \approx 0 + (x-1) - \frac{(x-1)^2}{2}.$$

Thus, we have that $\ln 1.1 \approx (0.1) - \frac{0.1^2}{2} = 0.095$.

6. Use the second order approximation to find $\sqrt{5}$.

Solution: We have that $\sqrt{4} = 2$ and 4 is close to 5 so we expand there. We have that

$$\sqrt{x} \approx 2 + \frac{x-4}{4} - \frac{1}{64}(x-4)^2.$$

Now we plug in 5 to get

$$\sqrt{5} \approx 2 + \frac{1}{4} - \frac{1}{64} \approx 2.234.$$

7. Use the second order approximation to find $e^{0.1}$.

Solution: We know that $e^0 = 1$ so we can expand around $x = 0$. Doing so gives

$$e^x \approx 1 + x + \frac{x^2}{2}.$$

Thus, we have that $e^{0.1} \approx 1 + 0.1 + 0.1^2/2 = 1.105$.

8. Use the second order approximation to find $\sec(0.1)$.

Solution: We know that $\sec(0) = 1/\cos(0) = 1$. We can expand there using the fact that the first derivative is $\sec(x)\tan(x)$ and the second derivative is $\sec(x)(\tan^2(x) + \sec^2(x))$. Thus, we get that the Taylor series is

$$\sec(x) \approx 1 + \frac{x^2}{2}.$$

Thus, we have that $\sec(0.1) \approx 1 + 0.1^2/2 = 1.005$.

9. Use the third order approximation to find $\sin(0.1)$.

Solution: We expand around 0 since $\sin 0 = 0$. We find that

$$\sin x \approx x - \frac{x^3}{6},$$

and so $\sin(0.1) \approx 0.1 - 0.1^3/6 = 0.0998$.

10. Use the second order approximation to find $\cos(0.1)$.

Solution: Expanding at $x = 0$ gives

$$\cos(x) \approx 1 - \frac{x^2}{2}.$$

Thus, $\cos(0.1) \approx 1 - 0.1^2/2 = 0.995$.